

Certainty Equivalence, Separation Principle, and Cooperative Output Regulation of Multi-Agent Systems by Distributed Observer Approach

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Abstract The cooperative output regulation problem of linear multi-agent systems was formulated and studied by the distributed observer approach in [20, 21]. Since then, several variants and extensions have been proposed, and the technique of the distributed observer has also been applied to such problems as formation, rendezvous, flocking, etc. In this chapter, we will first present a more general formulation of the cooperative output regulation problem for linear multi-agent systems that includes some existing versions of the cooperative output regulation problem as special cases. Then, we will describe a more general distributed observer. Finally, we will simplify the proof of the main results by more explicitly utilizing the separation principle and the certainty equivalence principle.

1 Introduction

The cooperative output regulation problem by distributed observer approach was first studied for linear multi-agent systems subject to static communication topology in [20], and then for linear multi-agent systems subject to dynamic communication topology in [21]. The problem is interesting because its formulation includes the leader-following consensus, synchronization or formation as special cases. In contrast with the output regulation problem of a single linear system [5, 8, 9], the information of the exogenous signal may not be available for every subsystem due to the communication constraints. Thus, information sharing, or, what is the same, cooperation among different subsystems is essential in the design of the control law. We call a control law that satisfies the communication constraints as a distributed

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control law. The core of the approach in [20, 21] is the employment of the so-called distributed observer, which provides the estimation of the leader's signal to each follower so that a distributed controller can be synthesized based on a purely decentralized controller and the distributed observer. Such an approach to designing a distributed controller is known as the certainty equivalence principle.

Since the publication of [20, 21], several variants and extensions of [20, 21] have been proposed [13, 14, 16, 19, 25]. The objectives of this chapter are three folds. First, we will present a more general formulation of the cooperative output regulation problem for linear multi-agent systems that includes some existing versions of the cooperative output regulation problem as special cases. Second, we will describe a more general distributed observer. Third, we will simplify the proof of the main results by more explicitly utilizing the separation principle and the certainty equivalence principle.

The cooperative output regulation problem by the distributed observer approach can also be generalized to some nonlinear systems such as multiple Euler-Lagrange systems [1] and multiple rigid-body systems [2]. Moreover, the distributed observer approach can also be applied to such problem as the leader-following flocking / rendezvous with connectivity preservation [6, 7].

It should be noted that the cooperative output regulation problem of multi-agent systems can also be handled by the distributed internal model approach [24, 27]. This approach has an additional advantage that it can tolerate perturbations of the plant parameters, and it does not need to solve the regulator equations. A combined distributed internal model and distributed observer approach is proposed in [14]. Nevertheless, in this chapter, we will only focus on the distributed observer approach.

Notation. \otimes denotes the Kronecker product of matrices. $\sigma(A)$ denotes the spectrum of a square matrix A . For matrices $x_i \in \mathbb{R}^{n_i \times p}$, $i = 1, \dots, m$, $\text{col}(x_1, \dots, x_m) = [x_1^T, \dots, x_m^T]^T$. We use $\sigma(t)$ to denote a piecewise constant switching signal $\sigma: [0, +\infty) \rightarrow \mathcal{P} = \{1, 2, \dots, \rho\}$ for some positive integer ρ where \mathcal{P} is called the switching index set. We assume the switching instants $t_0 = 0, t_1, t_2, \dots$ of σ satisfy $t_{k+1} - t_k \geq \tau > 0$ for some constant τ and for any $k \geq 0$, and τ is called the dwell time. I_n denotes the identity matrix of dimension n by n .

2 Linear Output Regulation

In this section, we review the linear output regulation problem for the class of linear time-invariant systems as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Ev(t), \quad x(0) = x_0, \quad t \geq 0, \\ y_m(t) &= C_m x(t) + D_m u(t) + F_m v(t), \\ e(t) &= Cx(t) + Du(t) + Fv(t), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $y_m \in \mathbb{R}^{p_m}$, $e \in \mathbb{R}^p$ and $u \in \mathbb{R}^m$ are the state, measurement output, error output, and input of the plant, and $v \in \mathbb{R}^q$ is the exogenous signal generated by an exosystem of the following form

$$\dot{v}(t) = Sv(t), \quad v(0) = v_0, \quad t \geq 0, \quad (2)$$

where S is some constant matrix.

Typically, the tracking error e is the difference between the system output y and the reference input r , i.e., $e = y - r$ where $y = Cx + Du + F_1v$ for some matrix F_1 and $r = F_2v$ for some matrix F_2 . Thus, in (1), we have $F = F_1 - F_2$. The tracking error e is assumed to be measurable, but it may not be the only measurable variable available for feedback control. Using the measurement output feedback control allows us to solve the output regulation problem for some systems which cannot be solved by the error output feedback control.

For convenience, we put the plant (1) and the exosystem (2) together into the following form

$$\begin{aligned} \dot{x} &= Ax + Bu + Ev, \\ \dot{v} &= Sv, \\ y_m &= C_mx + D_mu + F_mv, \\ e &= Cx + Du + Fv, \end{aligned} \quad (3)$$

and call (3) a composite system with $\text{col}(x, v)$ as the composite state.

In general, some components of the exogenous signal v , say, the reference inputs are measurable and some other components of the exogenous signal v , say, the unknown external disturbances are not measurable. Denote the unmeasured and measured components of v by $v_u \in \mathbb{R}^{q_u}$ and $v_m \in \mathbb{R}^{q_m}$, respectively, where $0 \leq q_u, q_m \leq q$ with $q_u + q_m = q$. Then, without loss of generality, we can assume v_u and v_m are generated, respectively, by the following systems:

$$\dot{v}_u = S_u v_u, \quad \dot{v}_m = S_m v_m, \quad (4)$$

for some constant matrices S_u and S_m . (4) is still in the form of (2) with $v = \text{col}(v_u, v_m)$ and $S = \text{block diag}[S_u, S_m]$.

To emphasize that v may contain both measurable and unmeasurable components, we can rewrite the plant (1) as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + E_u v_u(t) + E_m v_m(t), \quad x(0) = x_0, \quad t \geq 0, \\ y_m(t) &= C_m x(t) + D_m u(t) + F_{mu} v_u(t) + F_{mm} v_m(t), \\ e(t) &= Cx(t) + Du(t) + F_u v_u(t) + F_m v_m(t). \end{aligned} \quad (5)$$

where $E = [E_u \ E_m]$, $F_m = [F_{mu} \ F_{mm}]$, and $F = [F_u \ F_m]$.

We will consider the following so-called measurement output feedback control law:

$$\begin{aligned} u &= K_z z + K_y y_m, \\ \dot{z} &= \mathcal{G}_1 z + \mathcal{G}_2 y_m, \end{aligned} \quad (6)$$

where $z \in \mathbb{R}^{n_z}$ with n_z to be specified later, and $K_z \in \mathbb{R}^{m \times n_z}$, $K_y \in \mathbb{R}^{m \times p_m}$, $\mathcal{G}_1 \in \mathbb{R}^{n_z \times n_z}$, $\mathcal{G}_2 \in \mathbb{R}^{n_z \times p_m}$ are constant matrices.

This control law contains the following four types of control laws as special cases.

1. Full information when $y_m = \text{col}(x, v)$ and $n_z = 0$:

$$u = K_1 x + K_2 v, \quad (7)$$

where $K_1 \in \mathbb{R}^{m \times n}$ and $K_2 \in \mathbb{R}^{m \times q}$ are constant matrices.

2. (Strictly proper) Measurement output feedback when $K_y = 0$:

$$\begin{aligned} u &= K_z z, \\ \dot{z} &= \mathcal{G}_1 z + \mathcal{G}_2 y_m. \end{aligned} \quad (8)$$

3. Error output feedback when $y_m = e$ and $K_y = 0$:

$$\begin{aligned} u &= K_z z, \\ \dot{z} &= \mathcal{G}_1 z + \mathcal{G}_2 e. \end{aligned} \quad (9)$$

4. Combined error feedback and feedforward when $y_m = \text{col}(e, v)$:

$$\begin{aligned} u &= K_z z + K_v v, \\ \dot{z} &= \mathcal{G}_1 z + \mathcal{G}_e e + \mathcal{G}_v v, \end{aligned} \quad (10)$$

where $K_y = [0_{m \times p}, K_v]$, and $\mathcal{G}_2 = [\mathcal{G}_e, \mathcal{G}_v]$.

Needless to say that the control law (6) contains cases other than the control laws (7) to (10).

From the second equation of the plant (1) and the first equation of the control law (6), the control input u satisfies

$$u = K_z z + K_y (C_m x + D_m u + F_m v),$$

or

$$(I_m - K_y D_m) u = K_y C_m x + K_z z + K_y F_m v. \quad (11)$$

Therefore, the control law (6) is well defined if and only if $I_m - K_y D_m$ is nonsingular. It can be easily verified that the four control laws (7) to (10) all satisfy $K_y D_m = 0$. Thus, in what follows, we will assume $K_y D_m = 0$ though it suffices to assume $I_m - K_y D_m$ is nonsingular. As a result, we have

$$u = K_y C_m x + K_z z + K_y F_m v. \quad (12)$$

Under the assumption that $K_y D_m = 0$, the control law (6) can be put as follows:

$$\begin{aligned} u &= K_y C_m x + K_z z + K_y F_m v, \\ \dot{z} &= (\mathcal{G}_2(C_m + D_m K_y C_m))x + (\mathcal{G}_1 + \mathcal{G}_2 D_m K_z)z + \mathcal{G}_2(F_m + D_m K_y F_m)v. \end{aligned} \quad (13)$$

Thus, the closed-loop system composed of the plant (1) and the control law (6) can be put as follows:

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c v, \\ e &= C_c x_c + D_c v, \end{aligned} \quad (14)$$

where $x_c = \text{col}(x, z)$, and

$$\begin{aligned} A_c &= \begin{bmatrix} A + B K_y C_m & B K_z \\ \mathcal{G}_2(C_m + D_m K_y C_m) & \mathcal{G}_1 + \mathcal{G}_2 D_m K_z \end{bmatrix}, \quad B_c = \begin{bmatrix} E + B K_y F_m \\ \mathcal{G}_2(F_m + D_m K_y F_m) \end{bmatrix}, \\ C_c &= [C + D K_y C_m \quad D K_z], \quad D_c = F + D K_y F_m. \end{aligned}$$

In particular, for the full information control law (7), we have $y_m = \text{col}(x, v)$ and $n_z = 0$. Thus, $x_c = x$, $K_y C_m = K_1$, $K_y F_m = K_2$. As a result, we have

$$\begin{aligned} A_c &= A + B K_1, \quad B_c = E + B K_2, \\ C_c &= C + D K_1, \quad D_c = F + D K_2. \end{aligned}$$

We now describe the linear output regulation problem as follows.

Problem 1 *Given the plant (1) and the exosystem (2), find the control law of the form (6) such that the closed-loop system has the following properties.*

- **Property 1:** *The matrix A_c is Hurwitz, i.e., all the eigenvalues of A_c have negative real parts;*
- **Property 2:** *For any $x_c(0)$ and $v(0)$, $\lim_{t \rightarrow \infty} e(t) = 0$.*

At the outset, we list some standard assumptions needed for solving Problem 1.

Assumption 1 *S has no eigenvalues with negative real parts.*

Assumption 2 *The pair (A, B) is stabilizable.*

Assumption 3 *The pair $\left([C_m \ F_{mu}], \begin{bmatrix} A & E_u \\ 0 & S_u \end{bmatrix} \right)$ is detectable.*

Assumption 4 *The following linear equations*

$$\begin{aligned} X S &= A X + B U + E, \\ 0 &= C X + D U + F, \end{aligned} \quad (15)$$

admit a solution pair (X, U) .

Remark 1 Assumption 2 is made so that Property 1, that is, the exponential stability of A_c , can be achieved by the state feedback control. Assumption 3 together with Assumption 2 is to render the exponential stability of A_c by the measurement output feedback control. Assumption 1 is made only for convenience and loses no generality. In fact, if Assumption 1 is violated, then, without loss of generality, we can assume $S = \text{block diag } [S_1, S_2]$ where S_1 satisfies Assumption 1, and all the eigenvalues of S_2 have negative real parts. Thus, if a control law of the form (6) solves Problem 1 with the exosystem being given by $\dot{v}_1 = S_1 v_1$, then the same control law solves Problem 1 with the original exosystem $\dot{v} = Sv$. This is because Property 1 is guaranteed by Assumption 2 and/or Assumption 3, and, as long as the closed-loop system has Property 1, Property 2 will not be affected by exogenous signals that exponentially decay to zero.

Remark 2 Equations (15) are known as the regulator equations [8]. It will be shown in Theorem 2 and Remark 6 that, under Assumptions 1 to 3, Problem 1 is solvable by a control law of the form (6) only if the regulator equations are solvable. Moreover, if Problem 1 is solvable at all, necessarily, the trajectory of the closed-loop system starting from any initial condition is such that

$$\lim_{t \rightarrow \infty} (x(t) - Xv(t)) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} (u(t) - Uv(t)) = 0. \quad (16)$$

Therefore, Xv and Uv are the steady-state state and the steady-state input of the closed-loop system signal at which the tracking error e is identically zero. Thus, the steady state behavior of the closed-loop system is completely characterized by the solution of the regulator equations.

Remark 3 By Theorem 1.9 of [10], for any matrices E and F , the regulator equations (15) are solvable if and only if

$$\text{rank} \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} = n + p, \quad \forall \lambda \in \sigma(S). \quad (17)$$

Nevertheless, for a particular pair of (E, F) , the regulator equations may still have a solution even if (17) fails.

3 Solvability of the Linear Output Regulation Problem

In this section, we will study the solvability of Problem 1. Let us first present the following lemma on the closed-loop system.

Lemma 1 Suppose, under the control law (6), the closed-loop system (14) satisfies Property 1, i.e., A_c is Hurwitz. Then, the closed-loop system (14) also satisfies Property 2, that is, $\lim_{t \rightarrow \infty} e(t) = 0$, if there exists a matrix X_c that satisfies the following matrix equations:

$$\begin{aligned} X_c S &= A_c X_c + B_c, \\ 0 &= C_c X_c + D_c. \end{aligned} \quad (18)$$

Moreover, under the additional Assumption 1, the closed-loop system (14) also satisfies Property 2 only if there exists a unique matrix X_c that satisfies (18).

Remark 4 The proof is the same as that of Lemma 1.4 of [10], and is omitted. Here we only note that, if X_c satisfies (18), then the variable $\bar{x} = x_c - X_c v$ satisfies

$$\begin{aligned} \dot{\bar{x}} &= A_c \bar{x}, \\ e &= C_c \bar{x}. \end{aligned}$$

Since A_c is Hurwitz, $\lim_{t \rightarrow \infty} \bar{x}(t) = 0$, and hence, $\lim_{t \rightarrow \infty} e(t) = 0$. Since the solvability of the first equation of (18) is guaranteed as long as the eigenvalues of A_c do not coincide with those of S . Thus, Assumption 1 is not necessary for the sufficient part of Lemma 1. It suffices to require that the eigenvalues of A_c do not coincide with those of S .

Lemma 2 Under Assumption 1, suppose there exists a control law of the form (6) such that Property 1 holds. Then, Property 2 also holds if and only if there exist matrices X and U that satisfy the regulator equations

$$\begin{aligned} XS &= AX + BU + E, \\ 0 &= CX + DU + F. \end{aligned} \quad (19)$$

The proof is similar to that of Lemma 1.13 of [10] and is thus omitted.

Now let us first consider the full information case where the control law is defined by two constant matrices K_1 and K_2 . The two matrices K_1 and K_2 will be called the feedback gain and the feedforward gain, respectively.

Theorem 1 Under Assumption 2, let the feedback gain K_1 be such that $(A + BK_1)$ is Hurwitz. Then, Problem 1 is solvable by the full information control law (7) if Assumption 4 holds and the feedforward gain K_2 is given by

$$K_2 = U - K_1 X. \quad (20)$$

Proof. Under Assumption 2, there exists K_1 such that $A_c = A + BK_1$ is Hurwitz. Thus, under the control law (7), Property 1 is satisfied. Under Assumption 4, let $\bar{x} = x - Xv$ and K_2 be given by (20). Then we have

$$\begin{aligned} \dot{\bar{x}} &= (A + BK_1)\bar{x}, \\ e &= (C + DK_1)\bar{x}. \end{aligned} \quad (21)$$

Since $(A + BK_1)$ is Hurwitz, $\bar{x}(t)$ and hence $e(t)$ will approach zero as t tends to infinity. Thus, Property 2 is also satisfied.

Remark 5 By Lemma 2, Assumption 4 is also necessary for the solvability of Problem 1 by the full information control law (7) if Assumption 1 also holds.

We now turn to the construction of the measurement output feedback control law (6). Since we have already known how to synthesize a full information control law which takes the plant state x and the exosystem state v as its inputs, naturally, we seek to synthesize a measurement output feedback control law by estimating the state x and the unmeasurable exogenous signal v_u . To this end, lump the state x and the unmeasured exogenous signals v_u together to obtain the following system:

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{v}_u \end{bmatrix} &= \begin{bmatrix} A & E_u \\ 0 & S_u \end{bmatrix} \begin{bmatrix} x \\ v_u \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} E_m \\ 0 \end{bmatrix} v_m, \\ y_m &= [C_m \ F_{mu}] \begin{bmatrix} x \\ v_u \end{bmatrix} + D_m u + F_{mm} v_m. \end{aligned} \quad (22)$$

Employing the well known Luenberger observer theory suggests the following observer based control law:

$$\begin{aligned} u &= [K_1 \ K_{2u}]z + K_{2m}v_m, \\ \dot{z} &= \begin{bmatrix} A & E_u \\ 0 & S_u \end{bmatrix} z + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} E_m \\ 0 \end{bmatrix} v_m + L(y_m - [C_m \ F_{mu}]z - D_m u - F_{mm}v_m), \end{aligned} \quad (23)$$

where $K_{2u} \in \mathbb{R}^{m \times q_u}$, $K_{2m} \in \mathbb{R}^{m \times q_m}$, and $L \in \mathbb{R}^{(n+q_u) \times p_m}$ is an observer gain matrix.

The control law (23) can be put in the following form

$$\begin{aligned} u &= K_z z + K_{2m} v_m, \\ \dot{z} &= \mathcal{G}_1 z + \mathcal{G}_{21} y_m + \mathcal{G}_{22} v_m, \end{aligned} \quad (24)$$

where

$$\begin{aligned} K_z &= [K_1 \ K_{2u}], \\ \mathcal{G}_1 &= \begin{bmatrix} A & E_u \\ 0 & S_u \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} K_z - L([C_m \ F_{mu}] + D_m K_z), \\ \mathcal{G}_{21} &= L, \quad \mathcal{G}_{22} = \begin{bmatrix} B \\ 0 \end{bmatrix} K_{2m} + \begin{bmatrix} E_m \\ 0 \end{bmatrix} - L(F_{mm} + D_m K_{2m}). \end{aligned}$$

Since v_m is measurable, there exists a matrix C_v such that $v_m = C_v y_m$. Thus the control law (24) can be further put into the standard form (6) with $K_y = K_{2m} C_v$ and $\mathcal{G}_2 = \mathcal{G}_{21} + \mathcal{G}_{22} C_v$.

Theorem 2 *Under Assumptions 2 and 3, Problem 1 is solvable by the measurement output feedback control law (6) if Assumption 4 holds.*

Proof. First note that, by Assumption 2, there exists a state feedback gain K_1 such that $(A + BK_1)$ is Hurwitz, and, by Assumption 3, there exist matrices L_1 and L_2 such that

$$A_L = \begin{bmatrix} A & E_u \\ 0 & S_u \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} [C_m \ F_{mu}] = \begin{bmatrix} A - L_1 C_m & E_u - L_1 F_{mu} \\ -L_2 C_m & S_u - L_2 F_{mu} \end{bmatrix}$$

is Hurwitz. Let (X, U) satisfy the regulator equations, $L = \text{col}(L_1, L_2)$, and $K_2 = U - K_1 X$, and partition K_2 as $K_2 = [K_{2u}, K_{2m}]$. Let $\bar{x} = (x - Xv)$, $\bar{u} = (u - Uv)$, and $z_e = \begin{bmatrix} x \\ v_u \end{bmatrix} - z$. Then, it can be verified that

$$\begin{aligned} \bar{u} &= [K_1, K_{2u}] \begin{bmatrix} x \\ v_u \end{bmatrix} - K_z z_e + K_{2m} v_m - Uv \\ &= -K_z z_e + K_1 x + K_2 v - (K_2 + K_1 X)v \\ &= -K_z z_e + K_1 \bar{x}. \end{aligned}$$

In terms of \bar{x} and z_e , the closed-loop system is given by

$$\begin{aligned} \dot{\bar{x}} &= A\bar{x} + B\bar{u} = (A + BK_1)\bar{x} - BK_z z_e, \\ \dot{z}_e &= A_L z_e. \end{aligned} \tag{25}$$

Let A_c be the closed-loop system matrix. Then $\sigma(A_c) = \sigma(A + BK_1) \cup \sigma(A_L)$. Thus Property 1 is satisfied. To show $\lim_{t \rightarrow \infty} e(t) = 0$, first note that (25) implies that $\lim_{t \rightarrow \infty} \bar{x}(t) = 0$ and $\lim_{t \rightarrow \infty} z_e(t) = 0$. Then note that $e = Cx + Du + Fv = C(x - Xv) + D(u - Uv) + (CX + DU + F)v = C(x - Xv) + D(u - Uv) = (C + DK_1)\bar{x} - DK_z z_e$.

Remark 6 By Lemma 2, Assumption 4 is also necessary for the solvability of Problem 1 by a measurement output feedback control law of the form (6) if Assumption 1 also holds.

Specializing (24) to the two special cases with $v = v_u$ and $v = v_m$, respectively, gives the following two corollaries of Theorem 2.

Corollary 1 Under Assumptions 2 to 4 with $v_u = v$, Problem 1 is solvable by the following observer based control law:

$$\begin{aligned} u &= [K_1 \ K_2]z, \\ \dot{z} &= \begin{bmatrix} A & E \\ 0 & S \end{bmatrix} z + \begin{bmatrix} B \\ 0 \end{bmatrix} u + L(y_m - [C_m \ F_m]z - D_m u), \end{aligned} \tag{26}$$

where L is an observer gain matrix of dimension $(n + q)$ by p_m .

Corollary 2 Under Assumptions 2 to 4 with $v_m = v$, Problem 1 is solvable by the following observer based control law:

$$\begin{aligned} u &= K_1 z + K_2 v, \\ \dot{z} &= Az + Bu + Ev + L(y_m - C_m z - F_m v - D_m u), \end{aligned} \tag{27}$$

where L is an observer gain matrix of dimension n by p_m .

4 Linear multi-agent systems and distributed observer

In this section, we turn to the cooperative output regulation problem for a group of linear systems as follows:

$$\begin{aligned}\dot{x}_i &= A_i x_i + B_i u_i + E_i v, \\ y_{mi} &= C_{mi} x_i + D_{mi} u_i + F_{mi} v, \\ e_i &= C_i x_i + D_i u_i + F_i v, \quad i = 1, \dots, N,\end{aligned}\tag{28}$$

where $x_i \in \mathbb{R}^{n_i}$, $y_{mi} \in \mathbb{R}^{p_{mi}}$, $e_i \in \mathbb{R}^{p_i}$ and $u_i \in \mathbb{R}^{m_i}$ are the state, measurement output, error output, and input of the i th subsystem, and $v \in \mathbb{R}^q$ is the exogenous signal generated by a so-called exosystem as follows

$$\dot{v} = S v, \quad y_{m0} = C_0 v,\tag{29}$$

where $y_{m0} \in \mathbb{R}^{p_0}$ is the output of the exosystem.

Like the special case with $N = 1$, the exogenous signal v may also contain both unmeasured components $v_u \in \mathbb{R}^{q_u}$ and measured components $v_m \in \mathbb{R}^{q_m}$, where $0 \leq q_u, q_m \leq q$ with $q_u + q_m = q$. Then, like in (1), we can assume v_u and v_m are generated by (4). Correspondingly, we assume $C_0 = [0_{p_0 \times q_u}, C_{m0}]$ for some matrix $C_{m0} \in \mathbb{R}^{p_0 \times q_m}$. As a result, the plant (28) can be further written as follows.

$$\begin{aligned}\dot{x}_i &= A_i x_i + B_i u_i + E_{iu} v_u + E_{im} v_m, \\ y_{mi} &= C_{mi} x_i + D_{mi} u_i + F_{miu} v_u + F_{mim} v_m, \\ e_i &= C_i x_i + D_i u_i + F_{iu} v_u + F_{im} v_m, \quad i = 1, \dots, N,\end{aligned}\tag{30}$$

where $E_i = [E_{iu} \ E_{im}]$, $F_{mi} = [F_{miu} \ F_{mim}]$, and $F_i = [F_{iu} \ F_{im}]$.

Various assumptions are as follows.

Assumption 5 S has no eigenvalues with negative real parts.

Assumption 6 For $i = 1, \dots, N$, the pairs (A_i, B_i) are stabilizable.

Assumption 7 For $i = 1, \dots, N$, the pairs $\left([C_{mi} \ F_{miu}], \begin{bmatrix} A_i & E_{iu} \\ 0 & S_u \end{bmatrix}\right)$ are detectable.

Assumption 8 The linear matrix equations

$$\begin{aligned}X_i S &= A_i X_i + B_i U_i + E_i \\ 0 &= C_i X_i + D_i U_i + F_i\end{aligned} \quad i = 1, \dots, N,\tag{31}$$

have solution pairs (X_i, U_i) .

Remark 7 The system (28) is still in the form of (1) with $x = \text{col}(x_1, \dots, x_N)$, $u = \text{col}(u_1, \dots, u_N)$, $y_m = \text{col}(y_{m1}, \dots, y_{mN})$, $e = \text{col}(e_1, \dots, e_N)$. Thus, if the state v of the exosystem can be used by the control u_i of each follower, then, by Theorem 1, under Assumptions 6 and 8, the output regulation problem of the system (28) and the exosystem (29) can be solved by the following **full information control law**:

$$u_i = K_{1i}x_i + K_{2i}v, \quad i = 1, \dots, N, \quad (32)$$

where K_{1i} , $i = 1, \dots, N$, are such that $A_i + B_i K_{1i}$ are Hurwitz, and $K_{2i} = U_i - K_{1i}X_i$. Under the additional Assumption 7, there exist $L_i \in \mathbb{R}^{(n_i+q_u) \times p_{mi}}$ such that

$$\begin{bmatrix} A_i & E_{iu} \\ 0 & S_u \end{bmatrix} - L_i \begin{bmatrix} C_{mi} & F_{miu} \end{bmatrix}$$

are Hurwitz. Partition K_{2i} as $K_{2i} = [K_{2iu}, K_{2im}]$ with $K_{2iu} \in \mathbb{R}^{m_i \times q_u}$, $K_{2im} \in \mathbb{R}^{m_i \times q_m}$. Then, by Theorem 2, under Assumptions 6 to 8, the output regulation problem of the system (28) and the exosystem (29) can be solved by the following **measurement output feedback control law**:

$$\begin{aligned} u_i &= [K_{1i} \ K_{2iu}]z_i + K_{2im}v_m, \quad i = 1, \dots, N, \\ \dot{z}_i &= \begin{bmatrix} A_i & E_{iu} \\ 0 & S_u \end{bmatrix} z_i + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i + \begin{bmatrix} E_{im} \\ 0 \end{bmatrix} v_m \\ &\quad + L_i(y_{mi} - [C_{mi} \ F_{miu}]z_i - D_{mi}u_i - F_{mim}v_m). \end{aligned} \quad (33)$$

Nevertheless, in practice, the communication among different subsystems of (28) is subject to some constraints due to, say, the physical distance among these subsystems. Thus, the exogenous signal v or the measurable exogenous signal v_m may not be available for the control u_i of all the followers. Since, typically, $e_i = y_i - y_0$, the tracking error e_i may not be available for the control u_i of all the followers. To describe the communication constraints among various subsystems, we view the system (28) and the system (29) as a multi-agent system with (29) as the leader and the N subsystems of (28) as the followers, respectively. Let $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \bar{\mathcal{E}}_{\sigma(t)})^1$ with $\mathcal{V} = \{0, 1, \dots, N\}$ and $\bar{\mathcal{E}}_{\sigma(t)} \subseteq \mathcal{V} \times \mathcal{V}$ for all $t \geq 0$ be a switching graph, where the node 0 is associated with the leader system (29) and the node i , $i = 1, \dots, N$, is associated with the i th subsystem of the system (28). For $i = 0, 1, \dots, N$, $j = 1, \dots, N$, $(i, j) \in \bar{\mathcal{E}}_{\sigma(t)}$ if and only if u_j can use y_{mi} for control at time instant t . Let $\mathcal{N}_i(t) = \{j \mid (j, i) \in \bar{\mathcal{E}}_{\sigma(t)}\}$ denote the neighbor set of agent i at time instant t .

The case where the network topology is static can be viewed as a special case of switching network topology when the switching index set contains only one element. We will use the simplified notation \mathcal{G} to denote a static graph.

We will consider the following class of control laws.

$$\begin{aligned} u_i &= f_i(x_i, \xi_i), \quad i = 1, \dots, N, \\ \dot{\xi}_i &= g_i(\xi_i, y_{mi}, \xi_j, y_{mj}, j \in \mathcal{N}_i(t)), \end{aligned} \quad (34)$$

where both f_i and g_i are linear in their argument, and g_i is time-varying if the graph $\mathcal{G}_{\sigma(t)}$ is. It can be seen that, at each time $t \geq 0$, for any $i = 1, \dots, N$, u_i can make use of y_{m0} if and only if the leader is a neighbor of the subsystem i . Such a control law is called a **distributed control law**. If, for $i = 1, \dots, N$, f_i is independent

¹ See Appendix for a summary of graph.

of x_i , then the control law is called a **distributed measurement output feedback control law**. If, for $i = 1, \dots, N$, $\mathcal{N}_i(t) = \{0\}$, $\forall t \geq 0$, then the control law (34) is called a **purely decentralized control law**. In particular, (32) and (33) are called the **purely decentralized full information control law**, and the **purely decentralized measurement output feedback control law**.

We now describe our problem as follows.

Problem 2 *Given the systems (28), (29) and a switching graph $\bar{\mathcal{G}}_{\sigma(t)}$, find a distributed control law of the form (34) such that the closed-loop system has the following properties:*

- **Property 1:** *The origin of the closed-loop system with v being set to zero is asymptotically stable.*
- **Property 2:** *For any initial condition $x_i(0)$, $\xi_i(0)$, $i = 1, \dots, N$, and $v(0)$, the solution of the closed-loop system is such that*

$$\lim_{t \rightarrow \infty} e_i(t) = 0, \quad i = 1, \dots, N. \quad (35)$$

Clearly, the solvability of the above problem not only depends on the dynamics of the systems (28), (29), but also the property of the graph $\bar{\mathcal{G}}_{\sigma(t)}$. A typical assumption on the graph $\bar{\mathcal{G}}_{\sigma(t)}$ is as follows.

Assumption 9 *There exists a subsequence $\{i_k\}$ of $\{i : i = 0, 1, \dots\}$ with $t_{i_{k+1}} - t_{i_k} < v$ for some positive v such that every node $i = 1, \dots, N$ is reachable from the node 0 in the union graph $\bigcup_{j=i_k}^{i_{k+1}-1} \bar{\mathcal{G}}_{\sigma(t_j)}$.*

Remark 8 *Assumption 9 is similar to what was proposed in [12, 18, 21], and will be called **jointly connected** condition in the sequel. Since, under Assumption 9, the graph $\bar{\mathcal{G}}_{\sigma(t)}$ can be disconnected for all $t \geq 0$, it is perhaps the least stringent condition on the graph as opposed to some other conditions such as every time connected, or frequently connected. In particular, Assumption 9 is satisfied if Assumption 2 of [16] is. Thus the main result in [16] is essentially included in [21] even though the approach in [16] appears somehow different from that in [21].*

The static graph is a special case of the switching graph when $\rho = 1$. For this special case, Assumption 9 reduces to the following.

Assumption 10 *Every node $i = 1, \dots, N$ is reachable from the node 0 in the static graph $\bar{\mathcal{G}}$.*

Remark 9 *Let $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$ denote the subgraph of $\bar{\mathcal{G}}_{\sigma(t)}$ where $\mathcal{V} = \{1, \dots, N\}$, and $\mathcal{E}_{\sigma(t)} \subseteq \mathcal{V} \times \mathcal{V}$ is obtained from $\bar{\mathcal{E}}_{\sigma(t)}$ by removing all edges between the node 0 and the nodes in \mathcal{V} . Let $\tilde{\mathcal{A}}_{\sigma(t)} = [a_{ij}(t)]_{i,j=0}^N$ denote the weighted adjacency matrix of $\bar{\mathcal{G}}_{\sigma(t)}$, let $\mathcal{L}_{\sigma(t)}$ be the Laplacian matrix of $\mathcal{G}_{\sigma(t)}$ and $\Delta_{\sigma(t)} = \text{diag}(a_{10}(t), \dots, a_{N0}(t))$. Then, it is shown in Remark 14 of [22] that, under Assumption 9, the matrix $H_{\sigma(t)} = \mathcal{L}_{\sigma(t)} + \Delta_{\sigma(t)}$ has the property that all the eigenvalues of the matrix $\sum_{j=i_k}^{i_{k+1}-1} H_{\sigma(j)}$ have positive real parts. Furthermore, if the graph $\mathcal{G}_{\sigma(t)}$*

is undirected, then the matrix $\sum_{j=i_k}^{i_{k+1}-1} H_{\sigma(j)}$ is positive definite. In particular, under Assumption 10, the constant matrix $-H$ is Hurwitz.

5 Some Stability Results

As pointed out in Introduction, our approach is based on the employment of the distributed observer. To introduce the distributed observer. Let us first consider the stability property for the following class of switched linear systems:

$$\dot{x}(t) = (I_N \otimes A - \mu F_{\sigma(t)} \otimes (BK))x(t), \quad \sigma(t) \in \mathcal{P}, \quad (36)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $F_{\sigma(t)} \in \mathbb{R}^{n \times n}$ are given, and $\mu > 0$ and $K \in \mathbb{R}^{m \times n}$ are to be designed.

Assumption 11 *There exists a subsequence $\{i_k\}$ of $\{i : i = 0, 1, \dots\}$ with $t_{i_{k+1}} - t_{i_k} \leq v$ for some positive v such that all the eigenvalues of the matrix $\sum_{j=i_k}^{i_{k+1}-1} F_{\sigma(j)}$ have positive real parts.*

The stability property of the system of the form (36) has been extensively studied in the literature. We summarize the main results in two lemmas corresponding to the switching network and the static network, respectively, as follows.

Lemma 3 *Under Assumption 11, suppose the pair (A, B) is controllable. Then,*

(i) *If A is marginally stable, i.e., there exists a unique positive definite matrix P such that $PA + A^T P \leq 0$, and $F_{\sigma(t)}$ is symmetric, then, with $\mu = 1$, and $K = B^T P$, (36) is asymptotically stable;*

(ii) *If $B = I_n$ and A has no eigenvalues with positive real parts, then, with any $\mu > 0$, and $K = I_n$, (36) is asymptotically stable.*

Remark 10 *The stability property of the system of the form (36) was first studied in [22]. Part (i) of Lemma 3 was established in Theorem 1 of [22]. Part (ii) of Lemma 3 was established in Lemma 2 of [21]. As a corollary of Lemma 2 of [21], under Assumption 11, for any $\mu > 0$, the following system*

$$\dot{x} = -\mu F_{\sigma(t)} x \quad (37)$$

is asymptotically stable. As a special case of this result, when $\mathcal{P} = \{1\}$, the matrix $F_{\sigma(t)}$ is constant [11], and will be denoted by F . For this special case, the result of Lemma 3 can be strengthened to the following.

Lemma 4 *Under Assumption 11 with $\mathcal{P} = \{1\}$, suppose the pair (A, B) is stabilizable. Then,*

(i) *Let P be the unique positive definite matrix satisfying $PA + A^T P - PBB^T P + I_n \leq 0$, and $\mu \geq \delta^{-1}$ where $\delta = \min\{\text{Re}(\lambda_i(F))\}$. Then, with $K = B^T P$, (36) is asymptotically stable;*

(ii) If $B = I_n$, then, for any A , (36) is asymptotically stable with $K = I_n$, and sufficiently large μ .

Remark 11 The proof of Part (i) of Lemma 4 can be extracted from the proof of Theorem 2 of [24]. In fact, under Assumption 11 with $\mathcal{P} = \{1\}$, all the eigenvalues of F have positive real part. Let T be such that $TFT^{-1} = J$ is in the Jordan form of F . Denote the eigenvalues of F by $\lambda_1, \dots, \lambda_N$. Then $(I_N \otimes A) - \mu(F \otimes BK) = (T^{-1} \otimes I_n)((I_N \otimes A) - \mu(J \otimes BK))(T \otimes I_n)$. From the block triangular structure of J , we know that the eigenvalues of $(I_N \otimes A) - \mu(F \otimes BK)$ coincide with those of $A - \mu\lambda_i BK$, $i = 1, \dots, N$. Since the pair (A, B) is controllable, by Lemma 1 of [26], the algebraic Riccati equation

$$A^T P + PA - PBB^T P + I_n = 0 \quad (38)$$

admits a unique positive definite solution P . Moreover, for any $\mu \geq \delta^{-1}$ where $\delta = \min\{\operatorname{Re}(\lambda_i(F))\}$, the gain matrix $K = B^T P$ is such that $A - \mu\lambda_i BK$ and hence $(I_N \otimes A) - \mu(F \otimes BK)$ are Hurwitz.

Part (ii) of Lemma 4 was established in Theorem 1 of [20]. It is also a direct result of the fact that the eigenvalues of the matrix $(I_N \otimes A) - \mu(F \otimes I_n)$ are

$$\{\lambda_i(A) - \mu\lambda_j(F) : i = 1, \dots, n, j = 1, \dots, N\},$$

where $\lambda_i(A)$ and $\lambda_j(F)$ are the eigenvalues of A and F , respectively. Thus, the matrix $(I_N \otimes A) - \mu(F \otimes I_n)$ is Hurwitz for sufficiently large μ , and is Hurwitz for any positive μ if A does not have any eigenvalue with positive real part.

6 Solvability of the Cooperative Linear Output Regulation Problem

Given systems (28), (29) and the switching graph $\bar{\mathcal{G}}_{\sigma(t)}$ whose weighted adjacency matrix is denoted by $\bar{\mathcal{A}}_{\sigma(t)} = [a_{ij}(t)]_{i,j=0}^N$, we call the following compensator

$$\dot{\eta}_i = S_m \eta_i + \mu L_0 \left(\sum_{j \in \mathcal{N}_i(t)} a_{ij}(t) C_{m0} (\eta_j - \eta_i) \right), \quad i = 1, \dots, N, \quad (39)$$

where $\eta_0 = v_m$, $\mu > 0$ and $L_0 \in \mathbb{R}^{q_m \times p_0}$ are two design parameters, a distributed observer candidate for v_m , and call it a distributed observer for v_m if, for any $v_m(0)$ and $\eta_i(0)$, $i = 1, \dots, N$,

$$\lim_{t \rightarrow \infty} (\eta_i(t) - v_m(t)) = 0, \quad i = 1, \dots, N. \quad (40)$$

Whether or not (39) is a distributed observer of v_m depends on both the pair (C_{m0}, S_m) and the property of the graph.

Let $\tilde{\eta}_i = (\eta_i - v_m)$, and $\tilde{\eta} = \text{col}(\tilde{\eta}_1, \dots, \tilde{\eta}_N)$. Then, the system (39) can be put in the following compact form

$$\dot{\tilde{\eta}} = ((I_N \otimes S_m) - \mu(H_{\sigma(t)} \otimes L_0 C_{m0})) \tilde{\eta}. \quad (41)$$

Thus, the system (39) is a distributed observer of v_m if and only if the system (41) is asymptotically stable.

Remark 12 Since $((I_N \otimes S_m) - \mu(H_{\sigma(t)} \otimes L_0 C_{m0}))^T = (I_N \otimes S_m^T) - \mu(H_{\sigma(t)}^T \otimes C_{m0}^T L_0^T)$, system (41) is asymptotically stable if and only if a system of the form (36) with $A = S_m^T$, $F_{\sigma(t)} = H_{\sigma(t)}^T$, $B = C_{m0}^T$, and $K = L_0^T$ is asymptotically stable. Moreover, by Remark 9, under Assumption 9, all the eigenvalues of the matrix $\sum_{j=i_k}^{i_{k+1}-1} H_{\sigma(j)}$ have positive real parts. Furthermore, if the graph $\mathcal{G}_{\sigma(t)}$ is undirected, then the matrix $\sum_{j=i_k}^{i_{k+1}-1} H_{\sigma(j)}$ is positive definite.

Corresponding to the two purely decentralized control laws (32) and (33), we can synthesize two types of distributed control laws as follows:

1. Distributed full information control law:

$$\begin{aligned} u_i &= K_{1i}x_i + K_{2i}\eta_i, \quad i = 1, \dots, N, \\ \dot{\eta}_i &= S\eta_i + \mu L_0 \left(\sum_{j \in \mathcal{N}_i^-(t)} a_{ij}(t) C_0 (\eta_j - \eta_i) \right), \end{aligned} \quad (42)$$

where $K_{1i} \in \mathbb{R}^{m_i \times n_i}$ are such that $A_i + B_i K_{1i}$ are Hurwitz, $K_{2i} = U_i - K_{1i} X_i$, μ is some positive constant, and $\eta_0 = v = v_m$.

2. Distributed measurement output feedback control law:

$$\begin{aligned} u_i &= [K_{1i} \ K_{2iu}] z_i + K_{2im} \eta_i, \quad i = 1, \dots, N, \\ \dot{z}_i &= \begin{bmatrix} A_i & E_{iu} \\ 0 & S_u \end{bmatrix} z_i + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i + \begin{bmatrix} E_{im} \\ 0 \end{bmatrix} \eta_i \\ &\quad + L_i(y_{mi} - [C_{mi} \ F_{miu}] z_i - D_{mi} u_i - F_{mim} \eta_i), \\ \dot{\eta}_i &= S_m \eta_i + \mu L_0 \left(\sum_{j \in \mathcal{N}_i^-(t)} a_{ij}(t) C_{m0} (\eta_j - \eta_i) \right), \end{aligned} \quad (43)$$

where $\eta_0 = v_m$.

Remark 13 The control law (42) contains the distributed state feedback control law in [21] as a special case by letting $L_0 = C_0 = I_q$, or what is the same, $y_{m0} = v$, and the control law (43) contains the distributed measurement output feedback control law in [21] as a special case by letting $v_m = v$, and $L_0 = C_{m0} = I_q$.

Remark 14 Both of the control laws (42) and (43) are synthesized based on the certainty equivalence principle in the sense that they are obtained from the purely

decentralized control laws (32) and (33) by replacing v in (32) and v_m in (33) with η_i , respectively, where η_i is generated by a distributed observer of the form (39).

In [20] and [21], the solvability of the cooperative output regulation problem was established by means of Lemma 1, which incurred tedious matrix manipulation. In what follows, we will further simplify the proof of the solvability of the problem by means of the following Lemmas.

Lemma 5 *Consider the linear time-invariant system*

$$\dot{x} = Ax + Bu, \quad t \geq 0, \quad (44)$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ is Hurwitz, and $u \in \mathbb{R}^m$ is piecewise continuous in t and $\lim_{t \rightarrow \infty} u(t) = 0$. Then, for any initial condition $x(0)$, $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof. The conclusion follows directly from the fact that the system (44) is input-to-state stable with the input u decays to the origin asymptotically (Example 2.14 of [4]). A more elementary self-contained proof can be given as follows. For any $x(0)$, let

$$x(T) = e^{AT}x(0) + \int_0^T e^{A(T-\tau)}Bu(\tau)d\tau, \quad T \geq 0. \quad (45)$$

and

$$x(t) = e^{A(t-T)}x(T) + \int_T^t e^{A(t-\tau)}Bu(\tau)d\tau, \quad t \geq T. \quad (46)$$

It suffices to show $\lim_{t \rightarrow \infty} x(t) = 0$. Since A is Hurwitz, we have $\|e^{A(t-T)}\| \leq ke^{-\lambda(t-T)}$ for some $k > 0$ and $\lambda > 0$. Thus, for any $T \geq 0$, $\lim_{t \rightarrow \infty} \|e^{A(t-T)}x(T)\| = 0$. We only need to show that, for sufficiently large T , $\lim_{t \rightarrow \infty} \|\int_T^t e^{A(t-\tau)}Bu(\tau)d\tau\| = 0$. In fact, for any $t \geq T$,

$$\begin{aligned} \left\| \int_T^t e^{A(t-\tau)}Bu(\tau)d\tau \right\| &\leq \int_T^t ke^{-\lambda(t-\tau)}\|B\|\|u(\tau)\|d\tau \\ &\leq \frac{k\|B\|}{\lambda} \sup_{T \leq \tau \leq t} \|u(\tau)\| (1 - e^{-\lambda(t-T)}) \\ &\leq \frac{k\|B\|}{\lambda} \sup_{T \leq \tau \leq t} \|u(\tau)\|. \end{aligned} \quad (47)$$

Since $\lim_{t \rightarrow \infty} u(t) = 0$, for any $\varepsilon > 0$, there exists $T > 0$, such that, for any $t \geq T$, $\|u(t)\| \leq \frac{\lambda}{k\|B\|}\varepsilon$. Thus, (47) implies

$$\left\| \int_T^t e^{A(t-\tau)}Bu(\tau)d\tau \right\| \leq \varepsilon, \quad t \geq T. \quad (48)$$

Thus, $\lim_{t \rightarrow \infty} x(t) = 0$.

Lemma 6 *Under Assumption 1, suppose a control law of the form (6) solves the output regulation problem of the system (3). Let $\delta_u : [0, \infty) \rightarrow \mathbb{R}^m$ and $\delta_z : [0, \infty) \rightarrow \mathbb{R}^{n_z}$ be two piecewise continuous time functions such that $\lim_{t \rightarrow \infty} \delta_u(t) = 0$, and $\lim_{t \rightarrow \infty} \delta_z(t) = 0$. Then the following control law*

$$\begin{aligned} u &= K_z z + K_y y_m + \delta_u(t), \\ \dot{z} &= \mathcal{G}_1 z + \mathcal{G}_2 y_m + \delta_z(t), \end{aligned} \quad (49)$$

is such that $\lim_{t \rightarrow \infty} e(t) = 0$.

Proof. Denote the closed-loop system composed of (3) and (6) by (14). Then, A_c is Hurwitz. By Lemma 1, there exists a unique matrix X_c that satisfies equation (18). Let $\bar{x}_c(t) = x_c(t) - X_c v(t)$. Then the closed-loop system composed of (3) and (49) satisfies

$$\begin{aligned} \dot{\bar{x}}_c &= A_c \bar{x}_c + B_u \delta_u(t) + B_z \delta_z(t), \\ e &= C_c \bar{x}_c + D \delta_u(t), \end{aligned}$$

where $B_u = \text{col}(B, \mathcal{G}_2 D_m)$ and $B_z = \text{col}(0_{n \times n_z}, I_{n_z})$.

By Lemma 5, we have $\lim_{t \rightarrow \infty} \bar{x}_c(t) = 0$, and, thus, $\lim_{t \rightarrow \infty} e(t) = 0$.

From the proof of Lemma 6, we can immediately obtain the following result.

Corollary 3 *Under Assumption 1, suppose a control law of the form (6) solves Problem 1. Let $\mathcal{S}(t)$ be a piecewise continuous square matrix defined over $[0, \infty)$ such that $\dot{\eta} = \mathcal{S}(t)\eta$ is asymptotically stable, and K_u and K_z be any constant matrices. Then, under the following control law*

$$\begin{aligned} u &= K_z z + K_y y_m + K_u \eta, \\ \dot{z} &= \mathcal{G}_1 z + \mathcal{G}_2 y_m + K_z \eta, \\ \dot{\eta} &= \mathcal{S}(t)\eta, \end{aligned} \quad (50)$$

the closed-loop system also satisfies the two properties in Problem 1.

We now consider the solvability of Problem 2.

Lemma 7 *Suppose the distributed observer (41) is asymptotically stable. Then,*

(i) *Under Assumptions 5, 6, 8, Problem 2 is solved by the distributed full information control law (42);*

(ii) *Under the additional Assumption 7, Problem 2 is solved by the distributed measurement output feedback control law (43).*

Proof. Part (i) Let $K_{1i} \in \mathbb{R}^{m_i \times n_i}$ be such that $A_i + B_i K_{1i}$ are Hurwitz, and $K_{2i} = U_i - K_{1i} X_i$, $i = 1, \dots, N$. Then, by Remark 7, the purely decentralized full information control law (32) solves Problem 2. Since the control law (42) can be put in the following form:

$$\begin{aligned} u_i &= K_{1i}x_i + K_{2i}v + K_{2i}\tilde{\eta}_i, \quad i = 1, \dots, N, \\ \dot{\tilde{\eta}} &= ((I_N \otimes S) - \mu(H_{\sigma(t)} \otimes L_0 C_0)) \tilde{\eta}, \end{aligned} \quad (51)$$

where $\lim_{t \rightarrow \infty} \tilde{\eta}(t) = 0$. By Corollary 3, the proof is complete.

Part (ii) Under the additional Assumption 7, there exist $L_i \in \mathbb{R}^{(n_i+q_u) \times p_{mi}}$ such that

$$A_{Li} = \begin{bmatrix} A_i & E_{iu} \\ 0 & S_u \end{bmatrix} - L_i \begin{bmatrix} C_{mi} & F_{miu} \end{bmatrix}, \quad i = 1, \dots, N,$$

are Hurwitz. By Remark 7, Problem 2 can be solved by a control law of the form (33). Now denote the control law (33) by $u_i = k_i(z_i, v_m)$, $\dot{z}_i = g_i(z_i, k_i(z_i, v_m), y_{mi}, v_m)$, $i = 1, \dots, N$, and the control law (43) by

$$\begin{aligned} u_i &= k_i(z_i, \eta_i), \quad i = 1, \dots, N, \\ \dot{z}_i &= g_i(z_i, k_i(z_i, \eta_i), y_{mi}, \eta_i), \\ \dot{\tilde{\eta}} &= ((I_N \otimes S_m) - \mu(H_{\sigma(t)} \otimes L_0 C_{m0})) \tilde{\eta}. \end{aligned} \quad (52)$$

Then it is ready to verify that

$$k_i(z_i, \eta_i) = k_i(z_i, \tilde{\eta}_i + v_m) = k_i(z_i, v_m) + K_{2im}\tilde{\eta}_i, \quad (53)$$

and

$$\begin{aligned} &g_i(z_i, k_i(z_i, \eta_i), y_{mi}, \eta_i) \\ &= g_i(z_i, k_i(z_i, v_m) + K_{2im}\tilde{\eta}_i, y_{mi}, \tilde{\eta}_i + v_m) \\ &= g_i(z_i, k_i(z_i, v_m), y_{mi}, v_m) + \Gamma_i \tilde{\eta}_i, \end{aligned} \quad (54)$$

where

$$\Gamma_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix} K_{2im} + \begin{bmatrix} E_{im} \\ 0 \end{bmatrix} - L_i (F_{mim} + D_{mi} K_{2im}).$$

Since $\lim_{t \rightarrow \infty} \tilde{\eta}_i(t) = 0$ for $i = 1, \dots, N$, the proof follows from Corollary 3.

Remark 15 Lemma 7 is the reminiscent of the well known separation principle for the design of the Luenberger observer based output feedback control law. What is worth noting is that the closed-loop system is a time-varying system when the graph is a switching graph.

Combining Lemma 7 with Lemmas 3 and 4, respectively, leads to the following two theorems.

Theorem 3 Suppose Assumption 9 holds. Then,

(i) if the graph $\mathcal{G}_{\sigma(t)}$ is undirected, the pair (C_0, S) is observable, and S is marginally stable, then, under Assumptions 6 and 8, Problem 2 is solved by the distributed full information control law (42) with $\mu = 1$ and $L_0 = PC_0^T$ where P is the unique positive definite solution of the inequality $PS^T + SP \leq 0$, and, under the additional Assumption 7, Problem 2 is solved by the distributed measurement output feedback

control law (43) with $\mu = 1$ and $L_0 = PC_{m0}^T$ where P is the unique positive definite solution of the inequality $PS_m^T + S_m P \leq 0$;

(ii) if $y_{m0} = v$, and none of the eigenvalues of S has positive real part, then, under Assumptions 6 and 8, Problem 2 is solved by the distributed full information control law (42) for any $\mu > 0$ and $L_0 = I_q$; and

(iii) if $y_{m0} = v_m$, and none of the eigenvalues of S_m has positive real part, then, under Assumptions 6 to 8, Problem 2 is solved by the distributed dynamic measurement output feedback control law (43) for any $\mu > 0$ and $L_0 = I_{q_m}$.

Theorem 4 Suppose Assumption 10 holds. Let $\delta = \min\{\text{Re}(\lambda_i(H))\}$. Then,

(i) if the pair (C_0, S) is detectable, then, under Assumptions 5, 6 and 8, Problem 2 is solved by the distributed full information control law (42) with $\mu \geq \delta^{-1}$ and $L_0 = PC_0^T$ where P is the unique positive definite solution of the inequality $PS^T + SP - PC_0^T C_0 P + I_q \leq 0$, and, under the additional Assumption 7, Problem 2 is solved by the distributed measurement output feedback control law (43) with $\mu \geq \delta^{-1}$ and $L_0 = PC_{m0}^T$ where P is the unique positive definite solution of the inequality $PS_m^T + S_m P - PC_{m0}^T C_{m0} P + I_{q_m} \leq 0$;

(ii) if $y_{m0} = v$, then, under Assumptions 5, 6 and 8, Problem 2 is solved by the distributed full information control law (42) for sufficiently large μ , and $L_0 = I_q$; and

(iii) if $y_{m0} = v_m$, then, under Assumptions 5 to 8, Problem 2 is solved by the distributed measurement output feedback control law (43) for sufficiently large μ and $L_0 = I_{q_m}$.

Remark 16 In Parts (ii) and (iii) of Theorem 4, if none of the eigenvalues of S or S_m has positive real part, then μ can be any positive real number.

Remark 17 For the case where $v = v_m$, the control law (43) reduces to the following special form

$$\begin{aligned} u_i &= K_{1i} z_i + K_{2i} \eta_i, \quad i = 1, \dots, N, \\ \dot{z}_i &= A_i z_i + B_i u_i + E_i \eta_i + L_i (y_{mi} - C_{mi} z_i - D_{mi} u_i - F_{mi} \eta_i), \\ \dot{\eta}_i &= S \eta_i + \mu L_0 \left(\sum_{j \in \mathcal{N}_i^-(t)} a_{ij}(t) C_0 (\eta_j - \eta_i) \right), \end{aligned} \quad (55)$$

where $L_i \in \mathbb{R}^{n_i \times p_{mi}}$ are such that $(A_i - L_i C_{mi})$ are Hurwitz.

On the other hand, for the case where $v = v_u$, there is no measurable leader's signal v_m to estimate, the control law (43) reduces to the following special form

$$\begin{aligned} u_i &= [K_{1i} \ K_{2i}] z_i, \quad i = 1, \dots, N, \\ \dot{z}_i &= \begin{bmatrix} A_i & E_i \\ 0 & S \end{bmatrix} z_i + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i + L_i (y_{mi} - [C_{mi} \ F_{mi}] z_i - D_{mi} u_i). \end{aligned} \quad (56)$$

For this special case, the distributed observer is not needed and the control law is a purely decentralized one.

7 Some Variants and Extensions

In this section, we will make some remarks on some variants and extensions of the problem studied in this chapter.

7.1 Multiple leaders and containment control

The containment control problem involves multiple leaders and the asymptotic tracking of the output of followers to a convex hull of the state variables of the multiple leaders [17]. The problem formulation in Section 4 also includes the containment control problem as a special case by appropriately interpreting the leader system and the tracking error e_i . In fact, suppose there are multiple leaders of the following form:

$$\dot{v}_i = S_i v_i, \quad i = 1, \dots, l, \quad (57)$$

where, for $i = 1, \dots, l$, $v_i \in \mathbb{R}^{q_0}$ for some positive integer q_0 , and l is some integer greater than 1. Let $\text{Co} = \{\sum_{i=1}^l \alpha_i v_i, \alpha_i \geq 0, \sum_{i=1}^l \alpha_i = 1\}$. Then Co is called the convex hull of the points v_1, \dots, v_l . Let $v = \text{col}(v_1, \dots, v_l)$

Now define, for $i = 1, \dots, N$, the output of the subsystem i as $y_i = C_i x_i + D_i u_i + F_{1i} v$ for some matrix F_{1i} . Denote the reference input of each follower by $r = F_2 v$ where $F_2 = (\alpha_1, \dots, \alpha_l) \otimes I_{q_0}$. Let $e_i = y_i - r$. Then e_i is in the form given in (28) with $F_i = F_{1i} - F_2$. Finally, let $S = \text{block diag}[S_1, \dots, S_l]$, and $C_0 = F_2$. Then the multiple leader systems (57) can be put in the standard form (29).

It can be seen that the objective of making the tracking error e_i approach the origin asymptotically implies the asymptotic convergence of the output of all follower subsystems to the convex hull Co .

7.2 Local exogenous signals versus global exogenous signals

Another variant of the systems (28) is given as follows

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i + E_i v_i, \\ y_{mi} &= C_{mi} x_i + D_{mi} u_i + F_{mi} v_i, \\ e_i &= C_i x_i + D_i u_i + F_i v_i, \quad i = 1, \dots, N, \end{aligned} \quad (58)$$

where

$$\dot{v}_i = S_i v_i, \quad i = 1, \dots, N, \quad (59)$$

for some constant matrices S_i . This formulation is actually contained in (28) by defining $v = \text{col}(v_1, \dots, v_N)$, and $S = \text{block diag}[S_1, \dots, S_N]$ and redefining the matrices $E_i, F_{mi}, F_i, i = 1, \dots, N$.

7.3 Synchronized reference generator and the output synchronization

Given maps $\xi_i : [0, \infty) \rightarrow \mathbb{R}^p$ for $i = 1, \dots, N$ and a map $\bar{\xi} : [0, \infty) \rightarrow \mathbb{R}^p$, the elements of the set $\{\xi_i(\cdot) : i = 1, \dots, N\}$ are said to synchronize to $\bar{\xi}(\cdot)$ if $\lim_{t \rightarrow \infty} (\xi_i(t) - \bar{\xi}(t)) = 0$ for all $i = 1, \dots, N$, and are said to synchronize if they synchronize to some $\bar{\xi}(\cdot)$ [26].

Consider the following dynamic compensator

$$\dot{\eta}_i = S\eta_i + \mu L_0 \left(\sum_{j \in \mathcal{N}_i(t)} a_{ij}(t) C_0 (\eta_j - \eta_i) \right), \quad i = 1, \dots, N, \quad (60)$$

where $S \in \mathbb{R}^{q \times q}$, $C_0 \in \mathbb{R}^{p_0 \times q}$ are some given constant matrices, $\mathcal{N}_i(t)$ denote the neighbor set of the node i in the graph $\mathcal{G}_{\sigma(t)}$, and $\mu > 0$ and $L_0 \in \mathbb{R}^{q \times p_0}$ are to be designed.

The compensator (60) can be obtained from (39) by replacing $\tilde{\mathcal{N}}_i(t)$ by $\mathcal{N}_i(t)$.

We assume the graph $\mathcal{G}_{\sigma(t)}$ satisfies the following assumption.

Assumption 12 *There exists a subsequence $\{i_k\}$ of $\{i : i = 0, 1, \dots\}$ with $t_{i_{k+1}} - t_{i_k} \leq v$ for some positive v such that the union graph $\bigcup_{j=i_k}^{i_{k+1}-1} \mathcal{G}_{\sigma(t_j)}$ is connected.*

Then we have the following result.

Theorem 5 *Under Assumption 12,*

(i) *Suppose the graph $\mathcal{G}_{\sigma(t)}$ is undirected, S is marginally stable, and the pair (C_0, S) is observable. Then, with $\mu = 1$ and $L_0 = PC_0^T$ where the matrix P is the unique positive definite matrix such that $PS^T + SP \leq 0$, for any initial condition $\eta_i(0)$, $i = 1, \dots, N$, the solution of (60) is such that*

$$\lim_{t \rightarrow \infty} \left(\eta_i(t) - e^{St} \frac{\sum_{j=1}^N \eta_j(0)}{N} \right) = 0 \quad (61)$$

exponentially.

(ii) *If $C_0 = I_q$ and none of the eigenvalues of S has positive real part, then, with $L_0 = I_q$ and any $\mu > 0$, and any initial condition $\eta_i(0)$, $i = 1, \dots, N$, there exists some $\bar{\eta}_0 \in \mathbb{R}^q$ determined by $\eta_j(0)$, $j = 1, \dots, N$ such that*

$$\lim_{t \rightarrow \infty} (\eta_i(t) - e^{St} \bar{\eta}_0) = 0 \quad (62)$$

exponentially.

Remark 18 The proof of Part (i) of Theorem 5 can be extracted from the proof of Theorem 2 of [22]. Part (ii) of Theorem 5 was studied in Lemma 1 of [28] which is in turn based on the result in [15].

Remark 19 If the graph $\mathcal{G}_{\sigma(t)}$ is static and connected, and the pair (C_0, S) is detectable, then, Theorem 5 can be strengthened as follows. Let $\lambda_2(\mathcal{L})$ denote the smallest positive eigenvalue of the Laplacian matrix \mathcal{L} of the graph \mathcal{G} and P be the unique positive definite matrix such that $SP + PS^T - PC_0^T C_0 P + I_q \leq 0$. Then, for $\mu \geq \max\{1, \frac{1}{\lambda_2(\mathcal{L})}\}$ and $L_0 = PC_0^T$, the solution of (60) is such that

$$\lim_{t \rightarrow \infty} \left(\eta_i(t) - e^{St} \sum_{j=1}^N r_j \eta_j(0) \right) = 0, \quad (63)$$

where $r = \text{col}(r_1, \dots, r_N) \in \mathbb{R}^N$ is the unit vector such that $r^T \mathcal{L} = 0$. This special case of Theorem 5 is the direct result of Lemma 1 of [26].

Remark 20 The special case with $\mu = 1$, $C_0 = I_q$, and $L_0 = I_q$ of the dynamic compensator (60) was proposed in [29], and was called synchronized reference generators. Its main difference from the distributed observer (39) is that it does not contain a feedforward term $a_{i0}(v - \eta_i)$. If we define a virtual leader $\dot{v} = Sv$, and let $\tilde{\eta}_i = (\eta_i - v)$ and $\tilde{\eta} = \text{col}(\tilde{\eta}_1, \dots, \tilde{\eta}_N)$, then, the system (60) can be put in the following compact form

$$\dot{\tilde{\eta}} = (I_N \otimes S - \mu(L_{\sigma(t)} \otimes L_0 C_0)) \tilde{\eta}. \quad (64)$$

By Theorem 5, the compensator is not a distributed observer of the virtual leader $\dot{v} = Sv$ because, for $i = 1, \dots, N$, the convergence of η_i to v happens only if $\eta_i(0)$ and $v(0)$ satisfy some equality. As a result, the control law (42) with the distributed observer replaced by the synchronized reference generator (60) will not solve the output regulation problem of (28) with the exosystem being $\dot{v} = Sv$, $y_{m0} = C_0 v$. Nevertheless, by the same technique as used in the proof of Lemma 7, it is possible to show that this control law can still make the output y_i , $i = 1, \dots, N$, of (28) synchronize to a signal of the form $e^{St} \tilde{\eta}_0$ for some $\tilde{\eta}_0$ determined by $\eta_i(0)$, $i = 1, \dots, N$.

7.4 Discrete distributed observer

To introduce our problem, let \mathbb{Z}^+ denote the set of nonnegative integers, and $\sigma_d : \mathbb{Z}^+ \rightarrow \mathcal{P}$ where $\mathcal{P} = \{1, 2, \dots, p\}$ is a piecewise constant switching signal in the sense that there exists a subsequence t_i of \mathbb{Z}^+ , called switching instants, such that $\sigma_d(t) = p$ for some $p \in \mathcal{P}$ for $t_i \leq t < t_{i+1}$ for any $t_i \geq 0$ and all $t \in \mathbb{Z}^+$.

Consider the discrete-time counterpart of the linear system (36) of the following form

$$x(t+1) = (I_N \otimes A - \mu F_{\sigma_d(t)} \otimes (BK)) x(t), \quad t = 0, 1, \dots, \infty, \quad (65)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $F_{\sigma_d(t)} \in \mathbb{R}^{n \times n}$ is a piecewise switching matrix, and $\mu > 0$ and $K \in \mathbb{R}^{m \times n}$ are to be designed.

Assumption 13 (i) The switching times satisfy $t_{i+1} - t_i \geq \tau$ for some positive integer $\tau > 1$ for all t_i .

(ii) The matrix $F_{\sigma_d(t)}$ is symmetric for all $t \in \mathbb{Z}^+$, and there exists a subsequence $\{i_k\}$ of \mathbb{Z}^+ with $t_{i_{k+1}} - t_{i_k} \leq \nu$ for some positive ν such that all the eigenvalues of the matrix $\sum_{j=i_k}^{i_{k+1}-1} F_{\sigma_d(j)}$ have positive real parts.

Let \bar{A} be the real Jordan form of A , P be the nonsingular matrix such that $A = P^{-1}\bar{A}P$, and $\bar{B} = PB$. Then, we have the following result:

Lemma 8 Under Assumption 13, suppose all the eigenvalues of A are semi-simple with modulus 1, and the pair (A, B) is controllable. Then, with $K = B^T P^T P A$, the system (65) is asymptotically stable for all μ satisfying

$$0 < \mu \leq \min_{p \in \mathcal{P}} \left\{ \frac{1}{\|F_p \otimes (\bar{A}^T \bar{B} \bar{B}^T \bar{A})\|} \right\}.$$

Remark 21 Lemma 8 is taken from Lemma 3.1 of [23]. As pointed out in Remark 2.2 of [23], the assumption that all the eigenvalues of A are semi-simple with modulus 1 can be relaxed to the assumption that A is marginally stable, i.e., all the eigenvalues of A are inside the unit circle, and those eigenvalues of A with modulus 1 are semi-simple.

Now consider a discrete-time linear multi-agent system of the following form:

$$\begin{aligned} x_i(t+1) &= A_i x_i(t) + B_i u_i(t) + E_i v(t), \\ y_{mi}(t) &= C_{mi} x_i(t) + D_{mi} u_i(t) + F_{mi} v(t), \\ e_i(t) &= C_i x_i(t) + D_i u_i(t) + F_i v(t), \quad i = 1, \dots, N, \quad t = 0, 1, \dots, \infty, \end{aligned} \quad (66)$$

where $x_i \in \mathbb{R}^{n_i}$, $y_{mi} \in \mathbb{R}^{p_{mi}}$, $e_i \in \mathbb{R}^{p_i}$ and $u_i \in \mathbb{R}^{m_i}$ are the state, measurement output, error output, and input of the i th subsystem, and $v \in \mathbb{R}^q$ is the exogenous signal generated by a discrete-time exosystem as follows:

$$v(t+1) = S v(t), \quad y_{m0}(t) = C_0 v(t), \quad t = 0, 1, \dots, \infty, \quad (67)$$

where $S \in \mathbb{R}^{q \times q}$ is marginally stable. Let $\bar{\mathcal{G}}_{\sigma_d(t)}$ be a switching graph associated with (66) and (67) whose weighted adjacency matrix is denoted by $\bar{\mathcal{A}}_{\sigma_d(t)} = [a_{ij}(t)]_{i,j=0}^N$.

Define the following dynamic compensator

$$\eta_i(t+1) = S \eta_i(t) + \mu L_0 \left(\sum_{j \in \mathcal{N}_i(t)} a_{ij}(t) C_0 (\eta_j(t) - \eta_i(t)) \right), \quad i = 1, \dots, N, \quad (68)$$

where $\eta_0 = v$, and the scalar $\mu > 0$ and the matrix $L_0 \in \mathbb{R}^{q \times p_0}$ are to be designed. The system (68) is called a discrete distributed observer candidate for v , and is called

a discrete distributed observer for v if, for any $v(0)$ and $\eta_i(0)$,

$$\lim_{t \rightarrow \infty} (\eta_i(t) - v(t)) = 0, \quad i = 1, \dots, N. \quad (69)$$

Let $\tilde{\eta}_i = (\eta_i - v)$, and $\tilde{\eta} = \text{col}(\tilde{\eta}_1, \dots, \tilde{\eta}_N)$. Then, the system (68) can be put in the following compact form

$$\tilde{\eta}(t+1) = ((I_N \otimes S) - \mu(H_{\sigma_d(t)} \otimes L_0 C_0)) \tilde{\eta}(t), \quad t = 0, 1, \dots, \infty. \quad (70)$$

Since $((I_N \otimes S) - \mu(H_{\sigma_d(t)} \otimes L_0 C_0))^T = ((I_N \otimes S^T) - \mu(H_{\sigma_d(t)}^T \otimes L_0^T C_0^T))$, it is not difficult to deduce the conditions on various matrices and the graph for guaranteeing the stability property of (70) from Lemma 8. Consequently, the discrete counterparts of Theorems 3 and 4 can be obtained.

7.5 Distributed adaptive observer

A drawback of the distributed observer (39) is that the matrix S or S_m is used by the controller of every follower. A more realistic controller should only allow those followers who are the children of the leader to know the matrix S or S_m . In [3], assuming $y_{m0} = v$ and $S_m = S$, a distributed adaptive observer was proposed as follows:

$$\begin{aligned} \dot{S}_i &= \mu_1 \sum_{j=0}^N a_{ij}(t)(S_j - S_i), \quad i = 1, \dots, N, \\ \dot{\eta}_i &= S_i \eta_i + \mu_2 \sum_{j=0}^N a_{ij}(t)(\eta_j - \eta_i) \end{aligned} \quad (71)$$

where $S_i \in R^{q \times q}$, $\eta_i \in R^q$, $S_0 = S$, $\eta_0 = v$, $\mu_1, \mu_2 > 0$.

Moreover, the following result was established in [3]

Lemma 9 *Consider the system (71). Under Assumption 9, suppose all the eigenvalues of the matrix S are semi-simple with zero-real parts. Then, for any $\mu_1, \mu_2 > 0$ and for any initial condition $S_i(0)$, $\eta_i(0)$ and $v(0)$, for $i = 1, \dots, N$, $S_i(t)$ and $\eta_i(t)$ exist and are bounded for all $t \geq 0$, and*

$$\lim_{t \rightarrow \infty} (S_i(t) - S) = 0, \quad \lim_{t \rightarrow \infty} (\eta_i(t) - v(t)) = 0. \quad (72)$$

Remark 22 *Lemma 9 holds for any S if the graph is static and connected.*

8 Concluding Remarks

In this chapter, we have presented a unified framework for handling the cooperative output regulation problem of multi-agent systems using the distributed observer approach. The main result not only contains various versions of the cooperative output regulation problem for linear multi-agent systems in the literature as special cases, but also present a more general distributed observer. We have also simplified the proof of the main result by more explicitly utilizing the separation principle and the certainty equivalence principle. In summary, we conclude that, as long as a distributed observer exists, the cooperative output regulation problem of multi-agent systems is solvable if and only if the classical output regulation problem of each subsystem is solvable by the classical way as summarized in Section 3.

Acknowledgements This work has been supported in part by the Research Grants Council of the Hong Kong Special Administration Region under grant No. 412813, and in part by National Natural Science Foundation of China under grant No. 61174049.

Appendix

Appendix: Graph

A digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a finite set of nodes $\mathcal{V} = \{1, \dots, N\}$ and an edge set $\mathcal{E} = \{(i, j), i, j \in \mathcal{V}, i \neq j\}$. An edge from node i to node j is denoted by (i, j) . The node i is called the father of the node j and the node j the child of the node i . The node i is also called the neighbor of node j . If the digraph \mathcal{G} contains a sequence of edges of the form $\{(i_1, i_2), (i_2, i_3), \dots, (i_k, i_{k+1})\}$, then the set $\{(i_1, i_2), (i_2, i_3), \dots, (i_k, i_{k+1})\}$ is called a directed path of \mathcal{G} from i_1 to i_{k+1} , and node i_{k+1} is said to be reachable from node i_1 . A digraph is said to be connected if it has a node from which there exists a directed path to every other node. The edge (i, j) is called undirected if $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$. The digraph is called undirected if every edge in \mathcal{E} is undirected. A graph $\mathcal{G}_s = (\mathcal{V}_s, \mathcal{E}_s)$ is called a subgraph of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if $\mathcal{V}_s \subseteq \mathcal{V}$ and $\mathcal{E}_s \subseteq \mathcal{E} \cap (\mathcal{V}_s \times \mathcal{V}_s)$. Given a set of r graphs $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i)$, $i = 1, \dots, r$, the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{E} = \bigcup_{i=1}^r \mathcal{E}_i$ is called the union of graphs \mathcal{G}_i and is denoted by $\mathcal{G} = \bigcup_{i=1}^r \mathcal{G}_i$. The weighted adjacency matrix $\mathcal{A} = [a_{ij}]_{i,j=1}^N$ of \mathcal{G} is defined as $a_{ii} = 0$; for $i \neq j$, $a_{ij} > 0 \Leftrightarrow (j, i) \in \mathcal{E}$, and $a_{ij} = a_{ji}$ if the edge (j, i) is undirected. The Laplacian of \mathcal{G} is defined as $\mathcal{L} = [l_{ij}]_{i,j=1}^N$, where $l_{ii} = \sum_{j=1}^N a_{ij}$, $l_{ij} = -a_{ij}$ for $i \neq j$. To define a switching graph, let $\mathcal{P} = \{1, 2, \dots, \rho\}$ for some positive integer ρ . We call a time function $\sigma : [0, +\infty) \rightarrow \mathcal{P} = \{1, 2, \dots, \rho\}$ a piecewise constant switching signal if there exists a sequence $t_0 = 0 < t_1 < t_2, \dots$ satisfying $\lim_{i \rightarrow \infty} t_i = \infty$ such that, for any $k \geq 0$, for all $t \in [t_k, t_{k+1})$, $\sigma(t) = i$ for some $i \in \mathcal{P}$. \mathcal{P} is called the switching index set. Given a piecewise constant switching signal $\sigma : [0, +\infty) \rightarrow \mathcal{P} = \{1, 2, \dots, \rho\}$, and a set of ρ graphs $\mathcal{G}_i = (\mathcal{V}_i, \mathcal{E}_i)$,

$i = 1, \dots, \rho$ with the corresponding weighted adjacency matrices being denoted by \mathcal{A}_i , $i = 1, \dots, \rho$, we call a time-varying graph $\mathcal{G}_{\sigma(t)} = (\mathcal{V}, \mathcal{E}_{\sigma(t)})$ a switching graph with the weighted adjacency matrix $\mathcal{A}_{\sigma(t)}$ if, for any $k \geq 0$, for all $t \in [t_k, t_{k+1})$, $\mathcal{A}_{\sigma(t)} = \mathcal{A}_i$ for some $i \in \mathcal{P}$.

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